UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1202**
ASSESSMENT PATTERN	:	**MATH1202A**
MODULE NAME	:	**Algebra 2**
DATE	:	**29-Apr-09**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2008/09-MATH1202A-001-EXAM-203

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Give the definition of a *group*, defining the terms you use.

(b) Determine whether or not the following sets G under the given operation $\star$ form a group or not, justifying your answer:

(i) $G = \{x \in \mathbb{R} : x \geq 0\}$, $a \star b = +\sqrt{a^2 + b^2}$,

(ii) $G = \mathbb{R}$, $a \star b = \sqrt[3]{a^3 + b^3}$,

(iii) $G = \{x \in \mathbb{R} : x \neq -1\}$, $a \star b = 2a + 2b + 2ab + 2$,

(iv) $G = \mathbb{R}$, $a \star b = a + b + ab(a + b)$.

2. (a) Let G be a finite group and H a subgroup. Prove that $|H|$ divides $|G|$.

(b) State Fermat's Little Theorem.

(c) Find $\overline{3}^{358}$ in $\mathbb{Z}_{31}^*$.

(d) Find the solution to $x^{13} = \overline{2}$ in $\mathbb{Z}_{31}^*$.

3. (a) Let A be an $n \times n$ matrix. Give the definition of $\det(A)$. State, without proof, the effect on the determinant of each type of elementary row operation.

(b) Evaluate $\det \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

(c) Find $\det \begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{pmatrix}$, expressing your answer as a product of linear factors. Determine when the matrix is invertible.

4. (a) Let A be an $n \times n$ matrix over $\mathbb{R}$. Give the definition of:

(i) an *eigenvalue* λ of A;

(ii) an *eigenvector* $\mathbf{v}$ of A;

(iii) the *characteristic polynomial* $c_A(t)$ of A;

(iv) A is *diagonalizable* (over $\mathbb{R}$).

State the basic criterion for a matrix to be diagonalisable.

(b) Prove that if A has n distinct eigenvalues, then A is diagonalisable.

(c) Prove that if D is a diagonal matrix then $c_D(D) = 0$. Deduce that if A is diagonalisable then $c_A(A) = 0$. [Do not assume the Cayley-Hamilton Theorem.]

5. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

(i) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

(ii) Find A^n (for positive integers n).

(iii) Solve the system of differential equations

$$\begin{aligned} dx/dt &= x + y + z \\ dy/dt &= 2y + z \\ dz/dt &= 3z \end{aligned}$$

given that $x(0) = 1$, $y(0) = 0$ and $z(0) = 1$.

6. (a) Let A be a real symmetric matrix and let $\mathbf{u}$, $\mathbf{v}$ be eigenvectors associated to the eigenvalues λ and μ respectively, where $\lambda \neq \mu$. Prove that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors.

(b) Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

(c) Let A be a real matrix which is orthogonally diagonalisable. Prove that A is symmetric.

END OF PAPER